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Symmetries and first integrals of time-dependent higher-order constrained systems ★

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Abstract

Symmetries and dynamical symmetries of higher-order time-dependent constrained (degenerate) Lagrangian systems are investigated by methods of the calculus of variations on fibered manifolds. The First Theorem of Noether and its modifications are presented and the geometric interpretation of the arising conserved functions is clarified in terms of exterior differential systems. Symmetries of Lagrangians and Euler–Lagrange form, and dynamical symmetries of Poincaré–Cartan form θ_λ and the $d\theta_\lambda$ are studied and relations between these symmetries are found. In contrast to regular Lagrangian systems, constrained systems are shown to possess two by nature different kinds of conserved functions.

Keywords: Lagrangian; Euler–Lagrange equations; Poincaré–Cartan form; Lepagean 2-form; The first variation formula; Extremal; Hamilton extremal; Invariant transformation; Symmetry; Dynamical symmetry; Noetherian symmetry; First integral; Constant of the motion; Noether theorem; Constrained system

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1. Introduction

The fundamental Emmy Noether's paper [22] became a starting point for intensive studies of symmetries and first integrals of Lagrangian systems. By methods of differential geometry, which turned out to be the most effective tool in this field, a plenty of important results have been achieved. Foundations of the geometric theory of invariant variational

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problems have been laid by Trautman [29,30] who first applied modern geometric methods. His methods have been developed within the range of the calculus of variations on fibered manifolds by Krupka [11,12,14] who studied symmetries of Lagrangians, Euler–Lagrange forms and of solutions of the Euler–Lagrange equations, and provided a transparent geometric reformulation and generalization of the First Noether Theorem in a very general situation (of higher-order field theory). In the last 20 years, many authors have achieved important results namely in studies of symmetries and Noetherian symmetries of first and higher-order regular Lagrangian systems (see e.g. [23–26,4,5] and references therein); recently, also symmetries of degenerate first-order Lagrangians (constrained systems) have been intensively studied ([3,6,7,21] and others).

The purpose of this paper is to study symmetries of Lagrangian systems and the arising constants of the motion for the case of general (both regular and degenerate) time-dependent Lagrangians of any finite order on fibered manifolds (including the zero-order Lagrangian systems, i.e. Lagrangians linear in the velocities). Our approach is closely related with the above-mentioned work of Trautman and Krupka, and is based on our concept of *Lepagean 2-form* [16], and a geometric understanding of a regular, resp. constrained system in terms of the theory of exterior differential systems [16,17,19].

After recalling necessary prerequisites from the calculus on fibered manifolds (Section 2), we recall briefly the basic concepts of the geometric variational theory; for more details we refer e.g. to Saunders [27] and Krupka [11,12,14]. In Section 4 we recall the concept of a locally variational form [1,13,14], of a Lepagean 2-form = Lagrangian system [16,19] (which is a generalization of the concepts of *symplectic, presymplectic, cosymplectic and precosymplectic form* to any Lagrangian of any order), and of Euler–Lagrange equations and Hamilton equations of a Lagrangian system [11,12,9,15,16]. We also recall the geometric meaning of these equations as equations for integral sections of certain distribution, and a geometric definition of regularity according to [16,19]. Based on this interpretation of the solutions of the Euler–Lagrange and Hamilton equations, we discuss in Section 5 the concepts of constant of the motion and of a first integral and we show that they generally may differ if the Lagrangian system is not regular. This difference has to be taken into account namely if one is interested in using constants of the motion for learning the dynamics of constrained systems. Section 6 is a recapitulation of known results on symmetries of Lagrangians and Euler–Lagrange forms, and of the Noether theorem according to [11,12]. Section 7 deals with dynamical symmetries (i.e., defined on the phase space) of the Poincaré–Cartan form θ_λ and of Lepagean 2-form (in particular, of $d\theta_\lambda$). It contains a generalization of some results known from the regular first-order theory to the general case, namely, of theorems relating symmetries and first integrals (modifications of the First Noether Theorem), and brings their clear geometric interpretation. In Section 8 relations between symmetries of a Lagrangian, Euler–Lagrange form, Poincaré–Cartan form and Lepagean 2-form are clarified.

Throughout the paper we work with smooth finite-order manifolds and smooth (unless otherwise explicitly stated) mappings. We denote by T the tangent functor, by J^s the s-jet prolongation functor, * the pull-back, ∂_ξ the Lie derivative along a vector field ξ, by i_ξ the contraction by a vector field ξ, and by $[\cdot,\ \cdot]$ the Lie bracket of vector fields. We shall need the theory of distributions of nonconstant rank; at this point we refer e.g. to the appendix of the book by Libermann and Marle [20].

2. Calculus on fibered manifolds

A suitable geometric background for the study of time-dependent Lagrangian systems is a *fibered manifold* $\pi : Y \to X$ where the base X is a one-dimensional manifold and Y is a manifold of dimension $m+1$, and its *jet prolongations* $\pi_r : J^rY \to X, r \geq 1$, which enable one to consider velocities, accelerations, and other higher derivatives in a mathematically correct way. (In particular, one can take $X = \mathbb{R}$ or even $\pi : \mathbb{R} \times M \to \mathbb{R}$; in the latter case the prolongation $J^r(\mathbb{R} \times M)$ identifies naturally with $\mathbb{R} \times T^rM$. Throughout the paper, however, we shall work on a general fibered manifold over a one-dimensional base, since then many of the considerations and results can be directly transferred to field theory.) We denote for $r > k \geq 0$, by $\pi_{r,k} : J^rY \to J^kY$ the natural projection (here $J^0Y = Y$), which for every k, is a fibered manifold over the base J^kY. If (t, q^σ) is a fiber chart on Y, we denote by (t, q_i^σ), $1 \leq \sigma \leq m, 0 \leq i \leq r$, the associated chart on J^rY.

Recall that a vector field ξ on J^rY is called *π_r-projectable* if there exists a vector field ξ_0 on X such that $T\pi_r \cdot \xi = \xi_0$; ξ is called *π_r-vertical* if $T\pi_r \cdot \xi = 0$.

For a π-projectable vector field on Y one can define the *r-jet prolongation* $J^r\xi$ which is a vector field on J^rY; in fibered coordinates, where

$$\xi = \xi^0(t)\frac{\partial}{\partial t} + \xi^\sigma(t, q^\nu)\frac{\partial}{\partial q^\sigma},$$

one has

$$J^r\xi = \xi^0(t)\frac{\partial}{\partial t} + \xi^\sigma(t, q^\nu)\frac{\partial}{\partial q^\sigma} + \sum_{i=1}^{r} \xi_i^\sigma \frac{\partial}{\partial q_i^\sigma}, \tag{2.1}$$

where the functions ξ_i^σ, $1 \leq i \leq r$ are defined by the recurrent formula

$$\xi_i^\sigma = \frac{\mathrm{d}\xi_{i-1}^\sigma}{\mathrm{d}t} - q_i^\sigma \frac{\mathrm{d}\xi^0}{\mathrm{d}t}. \tag{2.2}$$

We shall need the concept of a *π_r-horizontal* 1*-form* ρ on J^rY; it is such that $\rho(\xi) = 0$ whenever its argument ξ is π_r-vertical. Clearly, a one form ρ on J^rY is π_r-horizontal iff in every fiber chart it is expressed in the following form:

$$\rho = f(t, q^\sigma, \ldots, q_r^\sigma)\,\mathrm{d}t$$

(i.e. it does not contain the differentials $\mathrm{d}q_i^\sigma$'s). To every 1-form ρ on J^rY one can assign a unique π_{r+1}-horizontal 1-form $h\rho$ on $J^{r+1}Y$, called the *horizontal part of* ρ; the mapping h is called the *π_r-horizontalization* and can be defined by the following formulas:

$$h\,\mathrm{d}t = \mathrm{d}t, \qquad h\,\mathrm{d}q^\sigma = q_1^\sigma \mathrm{d}t, \ \ldots, \ \mathrm{d}q_r^\sigma = q_{r+1}^\sigma\,\mathrm{d}t,$$

and

$$hf = f \circ \pi_{r+1,r}$$

for a (local) function f on J^rY. Recall that a 1-form ρ on J^rY is called *contact* if $h\rho = 0$. The distribution of 1-contact 1-forms on J^rY is generated by the following forms:

$$\omega_i^\sigma = \mathrm{d}q_i^\sigma - q_{i+1}^\sigma \,\mathrm{d}t\,, \quad 0 \le i \le r-1\,. \tag{2.3}$$

Note that the forms

$$\mathrm{d}t\,,\ \omega^\sigma, \ldots, \omega_{r-1}^\sigma\,,\ \mathrm{d}q_r^\sigma$$

form a local *basis of* 1*-forms* on J^rY. For fiber-coordinate expressions of differential forms on J^rY we shall often use this basis instead of the standard one $\mathrm{d}t, \mathrm{d}q_i^\sigma$, $1 \le i \le r$.

Clearly, every 1-form ρ on J^rY admits the unique decomposition into a sum of a horizontal form $h\rho$ and contact form $p\rho$, namely,

$$\pi_{r+1,r}^*\rho = h\rho + p\rho\,.$$

Similarly, a 2-form ρ on J^rY admits the unique decomposition into a sum of the so-called 1*-contact* form $p_1\rho$ and 2*-contact* form $p_2\rho$,

$$\pi_{r+1,r}^*\rho = p_1\rho + p_2\rho\,;$$

for our purpose it is sufficient to recall that in fibered coordinates the forms $p_1\rho$ and $p_2\rho$ are expressed as follows:

$$p_1\rho = \sum_{i=0}^{r} f_\sigma^i \omega_i^\sigma \wedge \mathrm{d}t\,, \qquad p_2\rho = \sum_{i,k=0}^{r} f_{\sigma\nu}^{ik} \omega_i^\sigma \wedge \omega_k^\nu\,.$$

3. The first variation formula

Let $\pi : Y \to X$ be a fixed fibered manifold.

Recall [11,12] that a *Lagrangian of order r* for π is a horizontal 1-form on J^rY. It is well known that to every Lagrangian λ there can be assigned a unique and globally defined one form θ_λ, called the *Lepagean* 1*-form* or *Poincaré–Cartan form* of λ; if λ is of order r then θ_λ is generally of order $2r-1$. The form

$$E_\lambda = p_1 \mathrm{d}\theta_\lambda \tag{3.1}$$

is then called the *Euler–Lagrange form* of the Lagrangian λ; for a Lagrangian of order r the Euler–Lagrange form is generally of order $2r$. In fibered coordinates, where

$$\lambda = L\,\mathrm{d}t \tag{3.2}$$

we have

$$\theta_\lambda = L\,\mathrm{d}t + \sum_{i=0}^{r-1} f_\sigma^{i+1}\omega_i^\sigma\,, \tag{3.3}$$

where

$$f_\sigma^{i+1} = \sum_{k=0}^{r-1-i} (-1)^k \frac{\mathrm{d}^k}{\mathrm{d}t^k} \frac{\partial L}{\partial q_{i+1+k}^\sigma}, \quad 0 \le i \le r-1, \tag{3.4}$$

and

$$E_\lambda = E_\sigma(L)\omega^\sigma \wedge \mathrm{d}t, \tag{3.5}$$

where the functions $E_\sigma(L)$, called the *Euler–Lagrange expressions*, have the familiar form

$$E_\sigma(L) = \sum_{k=0}^{r} (-1)^k \frac{\mathrm{d}^k}{\mathrm{d}t^k} \frac{\partial L}{\partial q_k^\sigma}, \quad 1 \le \sigma \le m. \tag{3.6}$$

Consider now a piece $\Omega \subset X$, and write (with an obvious convention) $\Omega = [a, b]$, where $[a, b]$ is a closed interval in $\mathbb{R}$, $a < b$. Denote by $\mathcal{S}_{[a,b]}(\pi)$ the set of sections γ of π such that the domain of γ is a neighborhood of $[a, b]$. If ρ is a 1-form on J^rY, recall [11,12] that the function

$$\rho_{[a,b]} : \mathcal{S}_{[a,b]}(\pi) \ni \gamma \to \int_a^b J^r\gamma^* \rho \in \mathbb{R}$$

is called the *variational function*, or the *action function* of ρ over $[a, b]$. If in particular, λ is a Lagrangian of order r, we get the *action function of the Lagrangian* λ

$$\lambda_{[a,b]} : \mathcal{S}_{[a,b]}(\pi) \ni \gamma \to \int_a^b J^r\gamma^* \lambda \in \mathbb{R}. \tag{3.7}$$

Note that since for a Lagrangian λ and its Lepagean 1-form θ_λ it holds

$$\int_a^b J^r\gamma^* \lambda = \int_a^b J^{2r-1}\gamma^* \theta_\lambda, \tag{3.8}$$

the action functions of λ and of θ_λ coincide.

Now, let us recall the concept of the first variation of the action function [11,12]. Let ξ be a π-projectable vector field on Y, ξ_0 its π-projection. Let $\{\phi_u\}$ (resp. $\{\phi_{0u}\}$) be the local 1-parameter group of ξ (resp. ξ_0). Let $\gamma \in \mathcal{S}_{[a,b]}(\pi)$ be a section. There exists an $\varepsilon > 0$ such that for each $u \in (-\varepsilon, \varepsilon)$ the section $\gamma_u = \phi_u\gamma\phi_{0u}^{-1}$ is defined in a neighborhood of $\phi_u([a, b])$. The 1-parameter family $\{\gamma_u\}$ of sections of π is called the *deformation of the section* γ *induced by* ξ. Now,

$$(-\varepsilon, \varepsilon) \ni u \to \lambda_{\phi_{0u}([a,b])}(\phi_u\gamma\phi_{0u}^{-1}) = \int_{\phi_{0u}([a,b])} J^r(\phi_u\gamma\phi_{0u}^{-1})^* \lambda \in \mathbb{R}$$

is a differentiable real-valued function of one real variable u; after a straightforward computation we get

$$\left(\frac{\mathrm{d}}{\mathrm{d}u}\lambda_{\phi_{0u}([a,b])}(\phi_u\gamma\phi_{0u}^{-1})\right)_{u=0}=\int_a^b J^r\gamma^*\,\partial_{J^r\xi}\lambda\,.$$

The arising action function of $\partial_{J^r\xi}\lambda$ over $[a, b]$, i.e.,

$$\partial_{J^r\xi}\lambda_{[a,b]}:\mathcal{S}_{[a,b]}(\pi)\ni\gamma\to\int_a^b J^r\gamma^*\,\partial_{J^r\xi}\lambda\in\mathbb{R} \tag{3.9}$$

is called the *first variation of the action function* $\lambda_{[a,b]}$, *induced by* ξ.

The following important assertion can be proved easily.

Theorem 3.1 (Krupka [11,12]). *Let* λ *be a Lagrangian of order* r, *let* θ_λ *be its Lepagean 1-form.*

(1) *For every* π*-projectable vector field* ξ *on* Y,

$$\partial_{J^r\xi}\lambda=h(i_{J^{2r-1}\xi}\,\mathrm{d}\theta_\lambda)+h(\mathrm{d}i_{J^{2r-1}\xi}\theta_\lambda)\,. \tag{3.10}$$

(2) *For every* π*-projectable vector field* ξ *on* Y *and every section* γ *of* π,

$$J^r\gamma^*\,\partial_{J^r\xi}\lambda=J^{2r-1}\gamma^*\,i_{J^{2r-1}\xi}\,\mathrm{d}\theta_\lambda+\mathrm{d}J^{2r-1}\gamma^*\,i_{J^{2r-1}\xi}\theta_\lambda\,. \tag{3.11}$$

(3) *For every* π*-projectable vector field* ξ *on* Y, *every closed interval* $[a, b]\in\mathbb{R}$ *and every section* γ *of* π,

$$\int_a^b J^r\gamma^*\,\partial_{J^r\xi}\lambda=\int_a^b J^{2r-1}\gamma^*\,i_{J^{2r-1}\xi}\,\mathrm{d}\theta_\lambda$$
$$+J^{2r-1}\gamma^*\,i_{J^{2r-1}\xi}\theta_\lambda(b)-J^{2r-1}\gamma^*\,i_{J^{2r-1}\xi}\theta_\lambda(a)\,. \tag{3.12}$$

Proof. Since for every 1-form ρ on J^rY,

$$\partial_{J^r\xi}h\rho=h\partial_{J^{r+1}\xi}\rho\,,$$

and $\lambda=h\theta_\lambda$, we get (1). Condition (2) is a consequence of (1) and the definition of the horizontalization mapping; (3) follows from (2). □

Formula (3.10) or (3.11) is called the *infinitesimal first variation formula*, (3.12) is called the *integral first variation formula*.

Notice that in terms of π*-vertical* vector fields, the infinitesimal first variation formula (3.10) takes the form

$$\partial_{J^r\xi}\lambda=i_{J^{2r}\xi}E_\lambda+h(\mathrm{d}i_{J^{2r-1}\xi}\theta_\lambda)\,,$$

where E_λ is the Euler–Lagrange form of λ.

Let λ be a Lagrangian of order r on π, consider the action function of λ over an interval $[a,b]$. A section $\gamma \in \mathcal{S}_{[a,b]}(\pi)$ is called a *critical section*, or an *extremal of* λ *on* $[a,b]$ if

$$\int_a^b J^r\gamma^* \partial_{J^r\xi}\lambda = 0 \tag{3.13}$$

for every π-vertical vector field ξ in a neighborhood of $\gamma([a,b])$ such that the support of ξ is a subset of $\pi^{-1}([a,b])$. γ is called an *extremal of* λ if it is an extremal of λ over any interval $[a,b]$.

By the first variation formula one has the following theorem.

Theorem 3.2 (Krupka [11,12]). *Let λ be a Lagrangian of order r, let γ be a section of π. The following conditions are equivalent:*

(1) *γ is an extremal of λ.*

(2) *For every π-projectable vector field ξ on Y,*

$$J^{2r-1}\gamma^* i_{J^{2r-1}\xi}\, \mathrm{d}\theta_\lambda = 0\,.$$

(3) *For every π-vertical vector field ξ on Y,*

$$J^{2r-1}\gamma^* i_{J^{2r-1}\xi}\, \mathrm{d}\theta_\lambda = 0\,.$$

(4) *In every fiber chart γ satisfies the system of ODE*

$$\left(\sum_{k=0}^{r}(-1)^k \frac{\mathrm{d}^k}{\mathrm{d}t^k}\frac{\partial L}{\partial q_k^\sigma}\right)\circ J^{2r}\gamma = 0, \quad 1 \le \sigma \le m\,.$$

Proof. Using the integral first variation formula we get immediately that (1) and (3) are equivalent.

Suppose (3). The form $\pi^*_{2r,2r-1}\mathrm{d}\theta_\lambda$ is decomposed into the 1-contact part (the Euler–Lagrange form E_λ), and the 2-contact part F_λ. Since the contraction of F_λ by a vertical vector field is a 1-contact form, it vanishes along $J^{2r}\gamma$. In this way we get

$$\begin{aligned} 0 &= J^{2r-1}\gamma^* i_{J^{2r-1}\xi}\,\mathrm{d}\theta_\lambda = J^{2r}\gamma^* i_{J^{2r}\xi}\,\pi_{2r,2r-1}\,\mathrm{d}\theta_\lambda \\ &= J^{2r}\gamma^* i_{J^{2r}\xi}\,E_\lambda + J^{2r}\gamma^* i_{J^{2r}\xi}\,F_\lambda = J^{2r}\gamma^* i_{J^{2r}\xi}\,E_\lambda. \end{aligned}$$

Denoting the components of E_λ by $E_\sigma(L)$, we get that $J^{2r}\gamma^*(E_\sigma(L)\xi^\sigma\,\mathrm{d}t) = 0$ for every π-vertical vector field ξ on Y, i.e., that $E_\sigma(L)\circ J^{2r}\gamma = 0$, proving (4).

Suppose (4). Then (by similar arguments as above) for every π-projectable vector field ξ on Y,

$$J^{2r-1}\gamma^* i_{J^{2r-1}\xi}\,\mathrm{d}\theta_\lambda = J^{2r}\gamma^* i_{J^{2r}\xi}\,E_\lambda = J^{2r}\gamma^*(\xi^\sigma\,\mathrm{d}t - \xi_0\,\mathrm{d}q^\sigma)(E_\sigma\circ J^{2r}\gamma) = 0\,,$$

proving (2).

Condition (3) follows from (2) trivially. □

The necessary and sufficient conditions for a section of π be an extremal of a Lagrangian λ are called the *Euler–Lagrange equations* of λ.

4. Time-dependent Lagrangian systems

Let $s \geq 1$ be an integer.

By a *dynamical form* of order s we mean a 1-contact 2-form E on J^sY such that $i_\xi E = 0$ for every $\pi_{s,0}$-vertical vector field ξ on J^sY; in fibered coordinates,

$$E = E_\sigma(t, q^\nu, \dots, q^\nu_s)\omega^\sigma \wedge \mathrm{d}t\,. \tag{4.1}$$

By a *solution* of a dynamical form E we mean a (local) section γ of π such that $E \circ J^s\gamma = 0$; equivalently,

$$E_\sigma(t, q^\nu, \dots, q^\nu_s) \circ J^s\gamma = 0\,, \quad 1 \leq \sigma \leq m\,.$$

Clearly, the concept of dynamical form on J^sY represents a "globalization" of the concept of a system of m ODE of order s to a fibered manifold. In particular, we shall be interested in the class of *variational* equations. Recall [11] that a dynamical form E on J^sY is said to be (*globally*) *variational* if there exists an integer r and a Lagrangian λ on J^rY such that (up to a projection) $E = E_\lambda$; E is said to be *locally variational* if there exists an open covering of J^sY such that E restricted to any element of this covering is a variational form. Note that the latter definition covers the important class of "global Euler–Lagrange equations to which there exists no global Lagrangian". The necessary and sufficient conditions of local variationality (generalizing the famous Helmholtz conditions) have been obtained in [31,13,1]; for the conditions of global variationality we refer e.g. to [1].

By definition, a *Lepagean 2-form* of order $s-1$ is a *closed* 2-form α on $J^{s-1}Y$ such that its 1-contact part $p_1\alpha$ is a dynamical form [16]. We have the following fundamental theorem clarifying the correspondence between variational equations and Lepagean 2-*forms*.

Theorem 4.1 ([16]). *Let E be a locally variational form of order s, $s \geq 1$. There exists a unique Lepagean 2-form α on $J^{s-1}Y$ such that $p_1\alpha = E$.*

Conversely, if α is a Lepagean 2-form then the form $E = p_1\alpha$ is locally variational.

Proof. Let $E = E_\sigma\, \mathrm{d}q^\sigma \wedge \mathrm{d}t$. One can check by a straightforward (but rather long) calculations that $\alpha = E + F$,

$$F = \sum_{i,k=0}^{s} F^{ik}_{\sigma\nu}\omega^\sigma_i \wedge \omega^\nu_k\,,$$

where

$$F^{jk}_{\sigma\nu} = -F^{kj}_{\nu\sigma}\,,$$
$$F^{jk}_{\sigma\nu} = \frac{1}{2}\sum_{l=0}^{s-j-k-1}(-1)^{j+l}\binom{j+l}{l}\frac{\mathrm{d}^l}{\mathrm{d}t^l}\frac{\partial E_\sigma}{\partial q^\nu_{j+k+l+1}}\,, \quad 0 \leq j+k \leq s-1\,,$$
$$F^{jk}_{\sigma\nu} = 0\,, \quad s \leq j+k \leq 2s-2\,,$$

and that α is projectable onto $J^{s-1}Y$ (for details see [14]). □

In accordance with Theorem 4.1, we define a *Lagrangian system* of order $s-1$ be a Lepagean 2-form α on $J^{s-1}Y$. (In what follows, we shall suppose that α is not projectable, i.e., that it depends on the q^{ν}_{s-1}'s nontrivially.) The manifold Y will be called the *configuration space* and the manifold $J^{s-1}Y$ the *phase space* [19]. Note that within this terminology, a Lagrangian system of order $s-1$ can be equivalently represented also by a locally variational form E of order s, or by the *class* of all equivalent (i.e., differing by a total derivative) local Lagrangians of different orders. It is known that every Lagrangian system possesses (local) Lagrangians of the minimal possible order c, where $c = \frac{1}{2}s$ in the case that s is even, and $c = \frac{1}{2}(s-1)$ if s is odd [31,16].

Remark 4.2. *Every Lepagean 2-form α can be locally expressed in the so-called canonical form*

$$\alpha = -\mathrm{d}H \wedge \mathrm{d}t + \sum_{i=0}^{s-c-1} \mathrm{d}p^i_\sigma \wedge \mathrm{d}q^\sigma_i ,$$

where $H, p_\sigma, \ldots, p^{s-c-1}_\sigma$ are certain (local) functions on the phase space, called a *Hamiltonian* and *momenta* of the Lagrangian system α, respectively [14]; if $\lambda_{\min} = L_{\min}\,\mathrm{d}t$ is a minimal-order Lagrangian then we can take

$$p^i_\sigma = (f_{\min})^{i+1}_\sigma , \qquad H = -L_{\min} + \sum_{i=0}^{s-c-1} p^i_\sigma q^\sigma_{i+1} ,$$

where the $(f_{\min})^{i+1}_\sigma$'s are given by (3.4).

Let α be a Lagrangian system of order $s-1$, $s \geq 1$. A (local) section γ of the configuration space is called an *extremal* of α if for every π-vertical vector field ξ on Y

$$J^{s-1}\gamma^* i_{J^{s-1}\xi}\alpha = 0 ; \tag{4.2}$$

a section δ of the phase space is called a *Hamilton extremal* of α if for every π_{s-1}-vertical vector field on $J^{s-1}Y$

$$\delta^* i_\xi \alpha = 0 . \tag{4.3}$$

Eq. (4.2) (resp. (4.3)) is called the *Euler–Lagrange equation* (resp. the *Hamilton equation*) of the Lagrangian system α.

We can see that Hamilton equation (4.3) is an equation for integral sections of a distribution on the phase space spanned by the (smooth) Pfaffian forms $i_\xi\alpha$ where ξ runs over all π_{s-1}-vertical vector fields on $J^{s-1}Y$; this distribution is called the *Euler–Lagrange distribution* [15,16]. The Euler–Lagrange distribution has an important subdistribution, the *characteristic distribution* of the closed 2-form α, spanned by the Pfaffian forms $i_\xi\alpha$ where ξ runs over the set of *all* vector fields on the phase space. We denote by Δ and $\mathcal{D}$ the Euler–Lagrange distribution and the characteristic distribution, respectively.

While the structure of Hamilton extremals is geometrically clear, the structure of extremals is not so transparent; however, one can see the following theorem.

Theorem 4.3 ([16,17]). *Let α be a Lagrangian system of order $s-1$, $s \geq 1$.*
(1) *If γ is an extremal of α then $J^{s-1}\gamma$ is an integral section of the characteristic distribution $\mathcal{D}$ of α.*
(2) *The set of extremals is in one-to-one correspondence with those Hamilton extremals which are of the form of prolongation (i.e., $\delta = J^{s-1}\gamma$).*

Proof.
(1) Let $J^{s-1}\gamma^* i_{J^{s-1}\xi}\alpha = 0$ for every *π-vertical* vector field ξ on Y. Computing this condition, we get $E \circ J^s\gamma = 0$, where $E = p_1\alpha$. Now, for *every* vector-field ξ on J^sY we have $0 = J^s\gamma^* i_\xi E = J^s\gamma^* i_\xi \pi^*_{s,s-1}\alpha$. This means that for every vector field ξ on $J^{s-1}Y$, $J^{s-1}\gamma^* i_\xi \alpha = 0$, proving that $J^{s-1}\gamma$ is an integral section of $\mathcal{D}$.
(2) If γ is an extremal then, by (1), $J^{s-1}\gamma$ is an integral section of $\mathcal{D}$, and hence of Δ. Conversely, if $J^{s-1}\gamma$ is a Hamilton extremal, then trivially, $J^{s-1}\gamma^* i_{J^{s-1}\xi}\alpha = 0$ for every π-vertical vector field on Y. □

It should be noted that in the particular case of $s = 1$ (first-order variational equations = zero-order Lagrangian systems, i.e., *Lagrangians linear in the velocities*) Hamilton extremals coincide with extremals; in other words, the Euler–Lagrange distribution (the characteristic distribution) describes the structure of *extremals*.

The above geometric interpretation of the dynamics of Lagrangian systems leads to the following geometric definition of regularity, and of a "constrained system" [16,19]. Namely, a Lagrangian system is called *regular* [16] if its characteristic distribution has a constant rank equal to one (this means that locally it can be spanned by one vector field). Since, moreover in this case extremals are in one-to-one correspondence with Hamilton extremals, regular Lagrangian systems possess the property that through every point in the phase space there passes exactly one maximal prolonged extremal (i.e., the motion is uniquely determined by the initial conditions).

Note that the above concept of regularity covers not only Lagrangians regular in the standard sense (the Hessian $\neq 0$), but also a wide class of Lagrangians which are usually considered as singular. Further note that our concept of regularity relates not to a particular Lagrangian, but to the whole *class of equivalent Lagrangians* (recently, regularity in this sense has been considered also in [8]).

A Lagrangian system is called *constrained* if it is not regular (i.e., if its Euler–Lagrange distribution is not of maximal rank on the phase space). A constrained Lagrangian system is called *semiregular* [17,19] if the rank of the Euler–Lagrange distribution Δ is (locally) constant, and $\Delta = \mathcal{D}$. More generally, we get a *geometric classification of Lagrangian systems according to the properties of their characteristic and Euler–Lagrange distribution*, i.e., according to their dynamical behavior [19]. The advantage of the geometric approach is a transparent and nonconfusing understanding of "constrained systems", of their dynamics, first integrals, etc. (cf. [17,19]).

5. Constants of the motion and first integrals of constrained systems

First, recall the notion of a first integral of a distribution. Let Δ be a distribution on a manifold M, locally spanned by a system of (continuous) Pfaffian forms. A function f, defined on an open subset of M, is called a *first integral* of Δ if the 1-form $\mathrm{d}f$ belongs to Δ (clearly, this means that $\mathrm{d}f$ can be locally expressed in terms of Pfaffian forms spanning Δ). If the rank of Δ is a constant then, equivalently, f is a first integral of Δ if $i_\xi\, \mathrm{d}f \equiv \partial_\xi f = 0$ for every vector field ξ belonging to Δ. Obviously, the geometric meaning of first integrals is the following: f is a first integral of Δ iff for every integral mapping φ of Δ,

$$\varphi^* \mathrm{d}f = \mathrm{d}(f \circ \varphi) = 0;$$

in other words, f is a constant along the integral mappings of Δ. First integrals of a distribution can be used to find integral manifolds. If, in particular, the distribution has constant rank and is completely integrable, then any system of independent first integrals $f_1, \dots, f_k$, where k = corank Δ, represents a complete (local) solution in an implicit form (more precisely, by the Frobenius theorem every system of such first integrals defines a chart adapted to Δ, i.e. the equations $f_1 = c_1$ (const.), $\dots$, $f_k = c_k$ (const.) are implicit equations for the integral mappings of Δ).

We have seen that the dynamics of a Lagrangian system is completely described by the Euler–Lagrange distribution Δ. Consequently, *if f is a first integral of the Euler–Lagrange distribution then f is constant along Hamilton extremals*, and in particular, *f is constant along extremals*. Similarly, *if f is a first integral of the characteristic distribution, it is constant along extremals*. Notice that since in general neither the set of Hamilton extremals nor the set of integral sections of $\mathcal{D}$ is in one-to-one correspondence with extremals, there can exist functions on the phase space which remain conserved along the (prolonged) extremals but are not first integrals of the characteristic distribution (i.e., $f \circ \delta$ = const. whenever $\delta = J^{s-1}\gamma$ but not necessarily for other Hamilton extremals). To distinguish between these two kinds of conserved functions we shall use the following terminology: A (local) function *f on the phase space* will be called a *constant of the motion* of an $(s-1)$th order Lagrangian system α if for every extremal γ

$$f \circ J^{s-1}\gamma = \text{const.}, \tag{5.1}$$

and a *first integral* of α if f is a first integral of the characteristic distribution.

Summarizing the results, we can see that *for regular Lagrangian systems* (i.e., such that the corresponding characteristic distribution is of rank one), *and for all zero-order Lagrangian systems* (i.e., Lagrangians linear in velocities) *the concepts of first integral and constant of the motion coincide. For constrained systems of order ≥ 1 they generally differ; every first integral is a constant of the motion but there can be constants of the motion which are not first integrals of the characteristic distribution.*

Remark 5.1. The dynamics of a *regular* Lagrangian system is (locally) represented by *one vector field* on the phase space, hence, constants of the motion/first integrals can be directly used to find the extremals (e.g. by applying the Liouville theorem).

On the other hand, the dynamical picture of *constrained systems* is much more complicated. If the Lagrangian system is *semiregular* then its Hamilton extremals are described by a system of involutive vector fields [17], and first integrals can be used to find the maximal integral manifolds of the system (e.g. by applying a generalized Liouville theorem [17,18]). Unfortunately, (prolonged) *extremals* of constrained systems *cannot* be described by means of a system of *involutive vector fields* on the phase space [19]. In this case, however, one can apply the so-called *constrained algorithm* [19] to describe the dynamics of any concrete constrained system. As a result, one gets a system of vector fields defined along certain submanifolds $\{M_\iota\}$ of the phase space. Now, one can use first integrals or constants of the motion to "strengthen" the vector fields on M_ι (for every index ι). It should be stressed, however, that often the vector fields along M_ι are noninvolutive; so the use of constants of the motion for practical integration of constrained systems is limited. (For details on the constrained algorithm and for examples of its application we refer to [19].)

6. The First Theorem of Emmy Noether

We start recalling the fundamental concepts of *invariant transformation* and *symmetry* of a differential form on a fibered manifold.

Consider a fibered manifold $\pi : Y \to X$. Let η be a p-form $p \geq 1$, on J^sY, $s \geq 0$. A diffeomorphism ϕ of J^sY is called an *invariant transformation* of η if

$$\phi^*\eta = \eta\,. \tag{6.1}$$

Now, let ξ be a vector field on J^sY, denote by $\{\phi_u^\xi\}$ its local 1-parameter group of transformations of J^sY. ξ is called a *symmetry* of η if for every u, the ϕ_u^ξ is an invariant transformation of η. Differentiating the equations $\phi_u^{\xi*}\eta = \eta$ with respect to the parameter u at $u = 0$, we get an *equivalent* condition for ξ be a symmetry of η, namely, a vector field ξ on J^sY is a *symmetry* of a p-form η on J^sY iff

$$\partial_\xi\eta = 0\,. \tag{6.2}$$

On fibered manifolds, one can consider different kinds of invariant transformations and symmetries. Recall that a diffeomorphism φ of the total space Y is called an *isomorphism* of the fibered manifold π if there exists a diffeomorphism φ_0 of the base X such that $\varphi_0 \circ \pi = \pi \circ \varphi$. In this case, for every $s > 0$ there arises a diffeomorphism $J^s\varphi$ of J^sY, the *s-jet prolongation of* φ. Hence, the group of invariant transformations of η has a subgroup of those invariant transformations which are of the form of *prolongation*. Similarly, a symmetry ξ of η can be of the form of *prolongation* of a projectable vector field ζ on Y. From the point of view of applications, the most interesting symmetries are those "living" either on Y, or on the phase space. In what follows, under a *symmetry* we shall mean a symmetry defined on Y, while a symmetry defined on the phase space we shall call *dynamical symmetry*. Note that if we have a differential form defined on the phase space $J^{s-1}Y$ and a (local) π-projectable vector field ξ on Y is its symmetry, then $J^{s-1}\xi$ is its dynamical symmetry; however, clearly not every dynamical symmetry has to correspond to a symmetry.

Let us turn to study symmetries of Lagrangians and Euler–Lagrange forms.

Let ϕ be an automorphism of the fibered manifold π. Let λ be a (local) Lagrangian of order r, E_λ its Euler–Lagrange form. ϕ is called an *invariant transformation* of the Lagrangian λ [29] if

$$J^r\phi^*\lambda = \lambda\,. \tag{6.3}$$

ϕ is called a *generalized invariant transformation* of the Lagrangian λ [29] if it is an invariant transformation of the Euler–Lagrange form of λ, i.e.,

$$J^{2r}\phi^* E_\lambda = E_\lambda\,. \tag{6.4}$$

We have the following fundamental lemma, relating the transformed Euler–Lagrange form of a Lagrangian with the Euler–Lagrange form of the transformed Lagrangian.

Theorem 6.1 (Krupka [12]). *Let ϕ be an automorphism of π, λ a Lagrangian on J^rY. Then the forms $J^{2r}\phi^* E_\lambda$ and $E_{J^r\phi^*\lambda}$ are defined on the same open subset of $J^{2r}Y$, and*

$$J^{2r}\phi^* E_\lambda = E_{J^r\phi^*\lambda}\,. \tag{6.5}$$

Proof. Denote θ_λ the Lepagean equivalent (Poincaré–Cartan form) of the Lagrangian λ. Then $J^{2r-1}\phi^*\theta_\lambda$ is the Lepagean equivalent of the Lagrangian $J^r\phi^*\lambda$ [12,13]. Now, for the Euler–Lagrange form of this Lagrangian we get

$$\begin{aligned} E_{J^r\phi^*\lambda} &= E_{J^{2r}\phi^* h\theta_\lambda} = E_{hJ^{2r-1}\phi^*\theta_\lambda} \\ &= p_1\,\mathrm{d}(J^{2r-1}\phi^*\theta_\lambda) = p_1(J^{2r-1}\phi^*\,\mathrm{d}\theta_\lambda) = J^{2r}\phi^*(p_1\,\mathrm{d}\theta_\lambda) = J^{2r}\phi^* E_\lambda \end{aligned}$$

(for details see [12] or [13]). □

The classes of invariant and generalized invariant transformations of a Lagrangian are related in the following way.

Theorem 6.2 (Krupka [12]). *Every invariant transformation of a Lagrangian λ is a generalized invariant transformation of λ.*

Proof. Proposition 6.2 immediately follows from Lemma 6.1. □

We can see, that a π-projectable vector field ξ on Y is a *symmetry of the Lagrangian* λ if and only if

$$\partial_{J^r\xi}\lambda = 0\,, \tag{6.6}$$

and ξ is a symmetry of the Euler–Lagrange form E_λ if and only if

$$\partial_{J^{2r}\xi} E_\lambda = 0\,. \tag{6.7}$$

Eq. (6.6) (resp. (6.7)) is called *Noether equation* (resp. *Noether–Bessel–Hagen equation*). Let us write these equations in fibered coordinates. Denote $\lambda = L\,\mathrm{d}t$, $E_\lambda = E_\sigma\,\mathrm{d}q^\sigma \wedge \mathrm{d}t$, and by (ξ_0, ξ_k^ν) the components of the prolongation of ξ (recall that $\xi_0 = \xi_0(t)$, $\xi^\nu = \xi^\nu(t, q^\sigma)$, and the ξ_k^ν, $k > 0$ are given by (2.2)). Then we get

$$\frac{\partial L}{\partial t}\xi_0 + \sum_{k=0}^{r} \frac{\partial L}{\partial q_k^\nu}\xi_k^\nu + L\frac{\mathrm{d}\xi_0}{\mathrm{d}t} = 0 \tag{6.8}$$

for the Noether equation, and

$$\frac{\partial E_\sigma}{\partial t}\xi_0 + \sum_{k=0}^{2r} \frac{\partial E_\sigma}{\partial q_k^\nu}\xi_k^\nu + E_\nu\frac{\partial \xi^\nu}{\partial q^\sigma} + E_\sigma\frac{\mathrm{d}\xi_0}{\mathrm{d}t} = 0 \tag{6.9}$$

for the Noether–Bessel–Hagen equation.

Note that to get a correct definition of symmetry of a Lagrangian, one must consider Lagrangian as a *differential form* λ, *not as a function* L. Working with Lagrangian functions (which, unfortunately, is very frequent in the existing literature) is ambiguous and can lead to confusion. In particular, for the *definition* of infinitesimal symmetry of a Lagrangian function Eq. (6.8) is used, making this concept geometrically mysterious (the Noether equation (6.8) surely does not mean that the *function* L is conserved along $J^r\xi$, but that $\partial_{J^r\xi}L + L\,\mathrm{d}\xi_0/\mathrm{d}t = 0$, where ξ_0 is the time-component of the vector field ξ).

Similarly, a correct and meaningful definition of symmetry of the Euler–Lagrange *equations* must be formulated for the corresponding Euler–Lagrange *differential* 2*-form*.

Remark 6.3. Noether equation and Noether–Bessel–Hagen equation can be used for important investigations in theoretical physics (in mechanics and in field theory). Namely, Noether equation can be viewed:

(1) as an equation for computing *symmetries* of a given Lagrangian; hence, in this case it is a PDE for vector fields ξ which leave the Lagrangian invariant;

(2) as a (system of) PDE for *Lagrangians* possessing prescribed symmetry (symmetries).

Similarly, Eq. (6.9) can be viewed not only as an equation for *symmetries* of a given locally variational form but also as an equation for *dynamical forms* possessing prescribed symmetries. In the latter case one can modify the problem as follows: find all dynamical forms which are locally variational and possess prescribed symmetries; then, of course, one has to solve (6.9) together with the (generalized) Helmholtz conditions with respect to the E_σ's; this point of view has been first applied in [26].

We have the following theorem on generalized invariant transformations.

Theorem 6.4 (Krupka [12]). *Let* ξ *be a* π*-projectable vector field on* Y, *let* λ *be a (local) Lagrangian of order* r, E_λ *its Euler–Lagrange form.* ξ *is a symmetry of* E_λ *if and only if one of the following equivalent conditions hold:*

(1) $E_{\partial_{J^r\xi}\lambda} = 0$,

(2) *there exists a unique closed* 1*-form* ρ *of order* $r-1$ *such that*

$$\partial_{J^r\xi}\lambda = h\rho\,.$$

Proof. Formula (6.5), applied to the local 1-parameter group of transformations of a projectable vector field ξ on Y, means that

$$\partial_{J^{2r}\xi} E_\lambda = E_{\partial_{J^r\xi}\lambda} . \tag{6.10}$$

Now, the first part of Theorem 6.4 trivially follows from (6.10). The equivalence of (1) and (2) becomes clear if one knows the kernel of the Euler–Lagrange mapping $\mathcal{E}$ (i.e., the mapping assigning to every Lagrangian its Euler–Lagrange form): it is known that a Lagrangian λ of order r belongs to $\ker \mathcal{E}$ if and only if $\lambda = h\rho$, where ρ is a closed 1-form of order $r - 1$ (for details see [11,12] or [14], where $\ker \mathcal{E}$ in the most general situation, i.e., higher-order field theory, is described). □

Corollary 6.5. *Every π-projectable vector field ξ which is a symmetry of a Lagrangian λ, is a symmetry of the Euler–Lagrange form E_λ of λ.*

Now, we can easily get the fundamental *First Theorem of Emmy Noether* as a direct consequence of the first variation formula (3.12) and the Noether equation (6.8).

Theorem 6.6 (Noether Theorem [22,11]). *Let λ be (local) a Lagrangian of order r (defined on an open subset $W \subset J^rY$), let θ_λ be its Lepagean equivalent. Let a π-projectable vector field ξ on Y be a symmetry of the Lagrangian λ. Let γ be an extremal of λ defined on $\pi_r(W) \subset X$. Then*

$$J^{2r-1}\gamma^* \mathrm{d} i_{J^{2r-1}\xi}\theta_\lambda = 0 . \tag{6.11}$$

Proof. By the integral first variation formula (3.12), over any interval $[a, b]$,

$$\int_a^b J^r\gamma^* \partial_{J^r\xi}\lambda = \int_a^b J^{2r-1}\gamma^* i_{J^{2r-1}\xi}\, \mathrm{d}\theta_\lambda + \int_a^b \mathrm{d}J^{2r-1}\gamma^* i_{J^{2r-1}\xi}\theta_\lambda .$$

Since γ is an extremal, by Theorem 3.2 the first term on the right-hand side of the above equation vanishes. By the invariance condition (6.6), the left-hand side equals zero, and we get

$$\int_a^b J^{2r-1}\gamma^*\, \mathrm{d}i_{J^{2r-1}\xi}\theta_\lambda = 0$$

over any interval $[a, b]$. Hence, (6.11) follows. □

Knowing a symmetry of λ, Noether Theorem provides us with the following function which *conserves along* $J^{2r-1}\gamma$:

$$f_{(L,\xi)} \equiv i_{J^{2r-1}\xi}\theta_\lambda = L\xi_0 + \sum_{i=0}^{r-1} f_\sigma^{i+1}(\xi_i^\sigma - q_{i+1}^\sigma \xi_0) , \tag{6.12}$$

where the f_σ^j's (resp. the ξ_i^σ's) are given by (3.4) (resp. (2.2)).

Note that for a Lagrangian of order r the function (6.12) is of order $2r - 1$. This means that, in general, from a Lagrangian of order greater than the minimal one, the Noether Theorem provides us with a function $f_{(L,\xi)}$ which lives higher than on the phase space, i.e., is not a constant of the motion. However, we have the following corollary.

Corollary 6.7. *Let λ be a minimal-order Lagrangian (of a Lagrangian system of order $s - 1$, $s \geq 1$). If a π-projectable vector field ξ on Y is a symmetry of λ then $i_{J^{s-1}\xi}\theta_\lambda$ is a constant of the motion.*

More generally, taking into account that for equivalent Lagrangians λ, λ', where $\lambda' = \lambda + h\,\mathrm{d}f$ one has $\theta_{\lambda'} = \theta_\lambda + \mathrm{d}f$, we get the following corollary.

Corollary 6.8. *Let λ' be a Lagrangian of order r for a Lagrangian system of order $s - 1$, $s \geq 1$, let $\lambda = \lambda' - h\,\mathrm{d}f$ be an equivalent Lagrangian such that θ_λ is of order $s - 1$. If a π-projectable vector field ξ on Y is a symmetry of λ', and $\partial_{J^r\xi} f =$ const., then $i_{J^{s-1}\xi}\theta_\lambda$ is a constant of the motion.*

Remark 6.9. Since

$$\int_a^b J^r\gamma^*\partial_{J^r\xi}\lambda = \int_a^b J^{2r-1}\gamma^*\partial_{J^{2r-1}\xi}\theta_\lambda\,,$$

Noether Theorem remains true if instead of a symmetry of the Lagrangian λ one takes a symmetry of its Lepagean equivalent θ_λ; *Let λ be a minimal-order Lagrangian (of a Lagrangian system of order $s - 1$, $s \geq 1$). If a π-projectable vector field ξ on Y is a symmetry of θ_λ then $i_{J^{s-1}\xi}\theta_\lambda$ is a constant of the motion.*

It remains to find the meaning of symmetries of a locally variational form E. Taking into account Theorem 3.2 and the first variation formula (similarly as in the proof of the Noether Theorem) we obtain a generalization of the Noether Theorem.

Theorem 6.10 ([11]). *Let E be a locally variational form of order s, let a π-projectable vector field ξ on Y be a symmetry of E. If λ is a (local) Lagrangian of order r for E on an open set W, and ρ is the unique closed 1-form of order $r - 1$ such that $\partial_{J^r\xi}\lambda = h\rho$, and if γ is an extremal of E defined on $\pi_r(W) \subset X$, then*

$$J^{2r-1}\gamma^*(\mathrm{d}i_{J^{2r-1}\xi}\theta_\lambda - \rho) = 0\,. \qquad (6.13)$$

Hence, a symmetry of E gives rise to a system of constants of the motion

$$i_{J^{s-1}\xi}\theta_\lambda - f\,,$$

where λ is a *minimal-order Lagrangian* for E and f is a (local) solution of the equation $\rho = \mathrm{d}f$.

7. Dynamical symmetries

In this section we shall study the problem how to find first integrals of the characteristic distribution $\mathcal{D}$ of a Lagrangian system. We shall show that first integrals are connected with dynamical symmetries of Lepagean 2-forms (roughly speaking, with dynamical symmetries of θ_λ and $d\theta_\lambda$); recall that we use the terminology "dynamical symmetries" for symmetries "living" in the phase space.

In the previous section we have obtained that every symmetry ξ on Y of the Lepagean equivalent θ_λ of a minimal-order Lagrangian λ gives rise to a *constant of the motion* $i_{J^{s-1}\xi}\theta_\lambda$ (Remark 6.9). It is easy to see, however, that a *stronger assertion* holds.

Theorem 7.1. *Let $s \geq 1$, let α be a Lagrangian system of order $s - 1$. Let ξ be a vector field on the phase space $J^{s-1}Y$.*

(1) *If ξ is a (dynamical) symmetry of α then the 1-form $i_\xi \alpha$ is closed; i.e., locally $i_\xi \alpha = df$, where f is a first integral of the characteristic distribution $\mathcal{D}$.*
(2) *If θ is a (local) 1-form of order $s - 1$ such that $\alpha = d\theta$, and if ξ is a (dynamical) symmetry of θ, then ξ is a (dynamical) symmetry of α and $i_\xi\theta$ is a first integral of $\mathcal{D}$.*

Remark 7.2. If the vector field $\partial/\partial t$ (the *time translation*) is a symmetry of a Lagrangian, i.e., if $\partial L/\partial t = 0$, then the Hamiltonian H is a constant of the motion. However, notice that the condition $\partial L/\partial t = 0$ implies that the vector field $\partial/\partial t$ is also a *symmetry of* $d\theta_\lambda$, which means that H *is a first integral.* In other words, for any (both regular and constrained) time-independent (= autonomous) Lagrangian system the Hamiltonian is conserved along *all* Hamilton extremals.

Similarly, if a *space translation* $\partial/\partial q^\nu$ is a symmetry of a (generally nonautonomous) Lagrangian, then the corresponding momentum p_ν is a constant of the motion. Moreover, from $\partial L/\partial q^\nu = 0$ we get $i_{\partial/\partial q^\nu} d\theta_\lambda = dp_\nu$, which means that the momentum p_ν *is a first integral.* Hence, if a Lagrangian system (regular or constrained) possesses a Lagrangian not depending on q^ν then the corresponding momentum is constant along the integral manifolds of the characteristic distribution.

In correspondence with [10,25], (dynamical) symmetries of a Lepagean 2-form will be called *Noetherian symmetries.*

Note that trivially, every vector field belonging to the characteristic distribution $\mathcal{D}$ is a Noetherian symmetry; however, symmetries of this kind are not very interesting, since they provide trivial first integrals.

Evidently, to a symmetry of α the corresponding first integral is unique up to a constant function. Conversely, given a first integral, the corresponding symmetry is not unique: More precisely, *if ξ is a Noetherian symmetry such that $i_\xi\alpha = df$, then for any vector field ζ belonging to $\mathcal{D}$, $\bar{\xi} = \xi + \zeta$ is another Noetherian symmetry satisfying $i_{\bar{\xi}}\alpha = df$. If ξ_1, ξ_2 are two symmetries such that $i_{\xi_1}\alpha = df = i_{\xi_2}\alpha$ then $\xi_1 - \xi_2$ belongs to $\mathcal{D}$.*

Let us note the following interesting property of Noetherian symmetries.

Theorem 7.3 ([18]). *Let ξ be a Noetherian symmetry of α. Let θ be defined on an open subset U of the phase space and such that $\mathrm{d}\theta = \alpha$ on U, and $\theta \notin \mathcal{D}$. Then in a neighborhood of every point $x \in U$ there is a Noetherian symmetry $\bar{\xi}$ such that $i_{\bar{\xi}}\theta = 0$. If $k \geq 0$ is an arbitrary but fixed integer and $\zeta_1, \ldots, \zeta_k$ belong to the characteristic distribution $\mathcal{D}$, defined in a neighborhood of x and such that $i_{\zeta_j}\theta \neq 0$, $1 \leq j \leq k$, then*

$$\bar{\xi} = \xi - \frac{1}{k}\sum_{j=1}^{k} \frac{i_\xi \theta}{i_{\zeta_j}\theta}\zeta_j \tag{7.1}$$

is a required symmetry.

Note that for every nontrivial Lagrangian system ($E \neq 0$) possessing at least one semispray belonging to $\mathcal{D}$, the requirement $\theta \notin \mathcal{D}$ of the above proposition is trivially satisfied for all Lepagean equivalents of minimal-order Lagrangians.

The following is a generalization of the classical *Poisson theorem* to higher-order non-regular Lagrangian systems.

Proposition 7.4. *The set of all Noetherian symmetries is a Lie algebra. If f_1, f_2 are first integrals of $\mathcal{D}$, and ξ_1, ξ_2 are some corresponding Noetherian symmetries, then*

$$g \equiv \{f_1, f_2\} = i_{\xi_1} i_{\xi_2}\alpha = i_{\xi_1}\mathrm{d}f_2 \tag{7.2}$$

is a first integral of $\mathcal{D}$, corresponding to the Noetherian symmetry $[\xi_1, \xi_2]$.

The first integral $\{f_1, f_2\}$ is called the *Poisson bracket* of the first integrals f_1, f_2.

Since for every C^1-vector field ζ belonging to $\mathcal{D}$ and every Noetherian symmetry ξ, $i_{[\xi,\zeta]}\alpha = \mathrm{d}i_\xi i_\zeta \alpha = 0$, we have the following corollary.

Corollary 7.5. *If ξ is a Noetherian symmetry and ζ is C^1-vector field belonging to $\mathcal{D}$ then the Lie bracket $[\xi, \zeta]$ belongs to $\mathcal{D}$.*

Note that in the case of *regular* Lagrangian systems, $\mathcal{D}$ is (locally) spanned by *one* vector field ζ (called *Hamiltonian vector field*, or *higher-order differential equation vector field*); then the above corollary takes the following form.

Let α be a regular Lagrangian system of order $s-1$, $s \geq 1$, let ζ be its Hamiltonian vector field and ξ its Noetherian symmetry. Then

$$[\xi, \zeta] = g\zeta \tag{7.3}$$

for a function g.

Remark 7.6. Authors studying symmetries of (in the standard sense) regular first-order Lagrangian systems often consider symmetries ξ satisfying the condition (7.3) (the so-called *generalized dynamical symmetries of ζ*). Clearly, Noetherian symmetries belong to this class of symmetries, but not every symmetry (7.3) is a Noetherian symmetry.

It is obvious how the concept of generalized dynamical symmetry is generalized to higher-order *regular* Lagrangian systems (including odd-order Euler–Lagrange equations). Notice, however, that it can be, in a meaningful way, further generalized only to those constrained systems whose dynamics is described by the characteristic distribution of a (locally) constant rank; among them, the most important are *semiregular* systems: If rank $\mathcal{D} = k$ then $\mathcal{D}$ is locally spanned by a system of $p = \operatorname{corank} \mathcal{D}$ vector fields $\zeta_1, \ldots, \zeta_p$ (of class at least C^1). Hence, we can say that a vector field ξ on the phase space is a *generalized dynamical symmetry of the characteristic distribution* $\mathcal{D}$, if for all $i = 1, \ldots, p$,

$$[\xi, \zeta_i] = g_i^j \zeta_j \tag{7.4}$$

for some functions g_i^j, $1 \le i, j \le p$.

For general constrained systems, one has either to consider symmetries of the Euler–Lagrange, resp. of the characteristic *distribution* instead of generalized dynamical symmetries or to compute the dynamical picture using the constrained algorithm and then to restrict the concept of generalized dynamical symmetry to a submanifold of the phase space along which the dynamics is described by a system of (at least C^1)-vector fields (cf. [19]).

8. Relations between various types of symmetries

Let us turn to a deeper study of relations between various types of symmetries.

Given a Lagrangian system, we have already seen that every symmetry of a Lagrangian λ is a symmetry of its Euler–Lagrange form E_λ (the converse being generally not true), and that every (dynamical) symmetry of a Lepagean equivalent θ_λ of a Lagrangian is a (dynamical) symmetry of the Lepagean 2-form α_{E_λ} associated with E_λ. Conversely, if ξ is a Noetherian symmetry of a Lagrangian system α and θ is a (local) 1-form such that $\alpha = \mathrm{d}\theta$ then ξ is not necessarily a dynamical symmetry of θ; we only get that $\partial_\xi \theta$ is closed.

Further the following proposition holds.

Proposition 8.1. *Let α be a Lagrangian system on $J^{s-1}Y$, let ξ be a (local) π-projectable vector field on Y.*

(1) *Let θ be a (local) 1-form such that $\alpha = \mathrm{d}\theta$. If ξ is a symmetry of θ then it is a symmetry of $\lambda = h\theta$.*

(2) *If ξ is a Noetherian symmetry then it is a symmetry of the locally variational form $E = p_1\alpha$.*

Proof. The first assertion follows from the first variation formula (3.10), which for any Lepagean 1-form θ of order r and its corresponding Lagrangian $h\theta$ reads

$$\partial_{J^{r+1}\xi} h\theta = h(\partial_{J^r\xi}\theta)\,.$$

We shall prove the second assertion. Let θ be a 1-form (defined on an open subset of $J^{s-1}Y$ where $J^{s-1}\xi$ is defined), and such that $\alpha = \mathrm{d}\theta$. By assumption, the form $i_{J^{s-1}\xi}\, \mathrm{d}\theta$

is closed. Hence the 1-form $\rho = \partial_{J^{s-1}\xi}\theta = i_{J^{s-1}\xi}\,\mathrm{d}\theta + \mathrm{d}i_{J^{s-1}\xi}\theta$ is closed, and for the Lagrangian $\lambda = h\theta$ we get

$$\partial_{J^s\xi}\lambda = h(\partial_{J^{s-1}\xi}\theta) = h\rho\,.$$

By Theorem 6.4, ξ is a (local) symmetry of $E_{h\theta} = E$. □

Summarizing the results, we have obtained the following relations between symmetries of a Lagrangian λ, its Poincaré–Cartan form θ_λ, the Euler–Lagrange form E_λ, and its associated Lepagean 2-form α_{E_λ}:

$$\begin{array}{ccc} \xi \text{ is a symmetry of } \theta_\lambda & \longrightarrow & \xi \text{ is a symmetry of } \lambda \\ \downarrow & & \downarrow \\ \xi \text{ is a symmetry of } \alpha_{E_\lambda} & \longrightarrow & \xi \text{ is a symmetry of } E_\lambda \end{array}$$

Remark 8.2. Note that some authors use a different terminology when considering symmetries of Lagrangian (or symplectic, resp. presymplectic) systems. For example, in [4,5] a vector field on $J^{2r-1}Y$ which is a symmetry of the Poincaré–Cartan form θ_λ is called a *Cartan symmetry*, and a vector field ξ on the configuration space Y such that $J^{2r-1}\xi$ is a Cartan symmetry is called *Noether symmetry*. From our considerations it is clear that Noether symmetries, if prolonged to the phase space, represent a kind of Noetherian symmetries in our sense, hence, they give rise to first integrals. On the other hand, Cartan symmetries give rise to first integrals of the characteristic distribution only if θ_λ is projectable onto the phase space.

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